

A criteria for a finite permutation group to be transitive

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Keywords:

Finite transitive permutation Groups

Abstract

Let G be a finite permutation group on a finite set Ω . The notion of G being quasi-transitive on Ω was defined by Alan Camina [2]; in that paper conditions were established that ensured a quasi-transitive group on a finite set Ω was transitive on Ω . The aim of this paper is to validate the conjecture made in [2]: given any group G , if G is quasi-transitive on a finite set Ω then G is transitive on Ω .

1 Introduction

Let G be a finite 2-transitive group acting on a finite set Ω ; a simple observation yields that for $G_{\alpha\beta}$ the pointwise stabiliser of $\alpha, \beta \in \Omega$, then $|G_{\alpha\beta}|$ constant for all $\{\alpha, \beta\} \subset \Omega$. Thus it is natural to ask, what does it mean for the action of G on Ω if $|G_{\alpha\beta}|$ is constant for all $\alpha, \beta \in \Omega$? Consider the case that $|G_{\alpha\beta}| = 1$ for all $\{\alpha, \beta\} \subset \Omega$; no non-trivial element in G fixes more than one point of Ω . Thus G must act Frobeniusly or regularly on each orbit of Ω ; however a finite group G can act Frobeniusly in at most one way and so G acts Frobeniusly on at most one orbit. Hence $|G_{\alpha\beta}| = 1$ for all $\{\alpha, \beta\} \subset \Omega$, if and only if G acts Frobeniusly on one orbit and regularly on the remaining orbits. From this we see that constant $|G_{\alpha\beta}|$ does not mean G has to be transitive; however if we assert that $|G_{\alpha\beta}| = t$ for all $\{\alpha, \beta\} \subset \Omega$ and $t > 1$, can we now conclude G has to act transitively? To study this hypothesis Camina made the following definition.

Definition 1.1. [2, Definition] *A finite permutation group G acting on a finite set Ω is called quasi-transitive if there is a natural number $t > 1$ such that $|G_{\alpha\beta}| = t$ for all two element subsets $\{\alpha, \beta\} \subset \Omega$.*

In [2], Camina established conditions for a group G that ensured quasi-transitive implies transitive.

Proposition 1.2. [2, Proposition 6] *If G is quasi-transitive and has a non-trivial normal soluble subgroup then G is transitive.*

Theorem 1.3. [2, Theorem] *If G is quasi-transitive and $G_{\alpha\beta}$ is abelian for all two-element subsets $\{\alpha, \beta\} \subset \Omega$, then G is transitive.*

In this note we validate the conjecture made in [2].

Theorem 1.4 (Main Theorem). *Let G be a finite permutation group G acting quasi-transitively on a finite set Ω , then G acts transitively on Ω .*

Our proof will make use of Proposition 1.2, Proposition 1.3 and a classification result by Bamberg, Giudici, Liebeck, Praeger and Saxl [1].

Theorem 1.5. [1, Theorem 1.1] *Every finite primitive $\frac{3}{2}$ -transitive group is either affine or almost simple.*

Theorem 1.6. [1, Theorem 1.2] Let G be a finite almost simple $\frac{3}{2}$ -transitive group of degree n on a set Ω . Then one of the following holds:

1. G is 2-transitive on Ω .
2. $n = 21$ and G is A_7 or S_7 acting on the set of pairs of elements of $\{1, \dots, 7\}$; the size of the non-trivial subdegrees is 10.
3. $n = \frac{1}{2}q(q-1)$, where $q = 2^f \geq 8$, and either $G = PSL_2(q)$ or $G = P\Gamma L_2(q)$ with f a prime; the size of the non-trivial subdegrees is $q+1$ or $f(q+1)$, respectively.

In particular, if we have a quasi-transitive group on a finite set Ω , and Δ is an orbit of G on which G acts as a finite almost simple $\frac{3}{2}$ -transitive group, then t is determined by Theorem 1.6.

2 The reduction

Before establishing the main theorem, we reduce the problem to the situation in Theorem 1.6. First we introduce some notation and recall some properties about quasi-transitive groups established in [2].

Assume a finite group G acts quasi-transitively but not transitively on a finite set Ω , and $|G_{\alpha\beta}| = t$ for all two element subsets $\{\alpha, \beta\}$ of Ω . As G is not acting transitively there are at least two orbits of G ; label the orbits for this action by $\Delta_1, \Delta_2, \dots, \Delta_r$ for some $r > 1$. For each $i = 1, \dots, r$ fix an element $\alpha_i \in \Delta_i$; the subdegrees of G on Δ_i are given by $d_i = |G_{\alpha_i}|/t$. As d_i is constant for the whole orbit Δ_i ; the group G acts $\frac{3}{2}$ -transitively on each Δ_i . In addition, by the Orbit-Stabiliser Theorem, $|G_{\alpha_i}| = \frac{|G|}{|\Delta_i|}$, thus $td_i = |G_{\alpha_i}| = \frac{|G|}{|\Delta_i|}$ and hence $d_1 \cdot |\Delta_1| = d_i \cdot |\Delta_i|$ for all $i = 1, \dots, r$. In [2] Camina established arithmetical conditions between the d_i and a condition for the action of G on Δ_i .

Lemma 2.1. [2, Lemma 1] Given d_i as above, the d_i 's are pairwise coprime.

Proposition 2.2. [2, Proposition 4] Let G be a quasi-transitive group then G acts faithfully on each orbit.

In particular none of the orbits Δ_i are a singleton element by Proposition 2.2. These two results have the following Corollary.

Corollary 2.3. Given Δ_i and d_i as above, $d_1 \cdot |\Delta_1| = d_i \cdot |\Delta_i|$ for all $i = 1, \dots, r$. In addition, the set of $|\Delta_i|$ are pairwise distinct.

Proof. The first result is discussed in the comments above. Hence it only remains to prove the second point.

Assume $|\Delta_i| = |\Delta_j|$, by the first part $d_i = d_j$ and hence $d_i = 1$ by Lemma 2.1. As $t = |G_{\alpha\beta}|$ for all two element subsets $\{\alpha, \beta\}$ of Ω , then $|G_{\alpha_i}| = |G_{\alpha_i\beta}|$ for all $\beta \in \Delta_i \setminus \{\alpha_i\}$; in particular $G_{\alpha_i} = G_{\alpha_i\beta}$. If $g \in G_{\alpha_i}$, then g fixes all of Δ_i ; however G acts faithfully on Δ_i by Proposition 2.2, thus $g = 1$ and G_{α_i} is trivial. By definition t must divide $|G_{\alpha_i}|$ and so $t = 1$ as well, contradicting that $t > 1$. \square

We now restrict a quasi-transitive group to Theorem 1.5.

Proposition 2.4. Let G be a finite quasi-transitive group on a finite set Ω with orbits $\Delta_1, \dots, \Delta_r$ and $r > 1$; then G acts as a finite primitive $\frac{3}{2}$ -transitive group on each Δ_i .

Proof. As $d_i = |G_{\alpha_i}|/t$, it is clear that $d_i = |\beta^{G_{\alpha_i}}|$ for any $\beta \in \Delta_i$; hence G is acting $\frac{3}{2}$ -transitive on each Δ_i . By a classical result about $\frac{3}{2}$ -transitive groups, G must act as a Frobenius or primitive group on each Δ_i [3, Theorem 10.4]. As G acts faithfully on Δ_i , if G also acts Frobeniusly then $t = |G_{\alpha_i\beta}| = 1$ for any $\beta \in \Delta_i$. Thus G must act primitively on each Δ_i . \square

Corollary 2.5. Let G be a finite quasi-transitive group on a finite set Ω with orbits $\Delta_1, \dots, \Delta_r$ and $r > 1$; then G acts as a almost simple $\frac{3}{2}$ -transitive group on each Δ_i .

Proof. G acts as a primitive $\frac{3}{2}$ -transitive group on each Δ_i by Proposition 2.4. Hence G must be either an affine group or an almost simple group by Theorem 1.5. Affine groups are characterised by having a unique minimal normal subgroup which is elementary abelian, in which case G is transitive on Ω by Theorem 1.3. Thus G must be almost simple. \square

3 The main theorem

In the previous section we established that if a finite group G acts quasi-transitively on a finite subset Ω with orbits $\Delta_1, \dots, \Delta_r$ and $r > 1$; then G must act as an almost simple $\frac{3}{2}$ -transitive group on each Δ_i . In particular, it must be one of the groups given in Theorem 1.6:

1. G is 2-transitive on Ω .
2. $n = 21$ and G is A_7 or S_7 acting on the set of pairs of elements of $\{1, \dots, 7\}$; the size of the nontrivial subdegrees is 10.
3. $n = \frac{1}{2}q(q-1)$, where $q = 2^f \geq 8$, and either $G = PSL_2(q)$ or $G = P\Gamma L_2(q)$ with f a prime; the size of the nontrivial subdegrees is $q+1$ or $f(q+1)$, respectively.

As $r > 1$, the group G must act on each orbit as one of the above group actions. Choose two orbits, say Δ_1 and Δ_2 ; it is enough to show that G can not act on both of these orbits via any combination of the actions given above.

Step 1: G does not act as a 2-transitive group on both Δ_1 and Δ_2 If G acts 2-transitively on Δ_i , then by applying the Orbit-Stabiliser Theorem first to G and then to G_{α_i} , we obtain $|G| = |\Delta_i|(|\Delta_i| - 1) \cdot t$. Hence if G is 2-transitive on Δ_i , then $|\Delta_i|$ must solve the quadratic equation $tx^2 - tx - |G| = 0$; however this equation can have at most one positive integer solution. G can be 2-transitive on at most one orbit because the $|\Delta_i|$ are pairwise distinct by Corollary 2.3.

Step 2: G must act as a 2-transitive group on one of Δ_1 and Δ_2 It is enough to show that G can not act as in (2) or (3) on the two orbits. However in both lists, the isomorphism type of G is associated to a unique action. Hence G can only act on one orbit as a group in (2) or (3); thus must act on the other orbit as a 2-transitive group.

In fact we have established that a quasi-transitive group which is not transitive must have exactly 2 orbits.

Step 3: G can not act on Δ_i as in (3) Assume G acts on Δ_1 as in (3); then G is either $PSL_2(q)$ or $P\Gamma L_2(q)$ for $q = 2^f \geq 8$. Also $|\Delta_1| = \frac{1}{2}q(q-1)$ and $d_1 = q+1$ or $f(q+1)$ respectively. As $|G_{\alpha\beta}| = t = \frac{|G|}{|\Delta_1| \cdot d_1}$, it follows that $t = 2$ and $G_{\alpha\beta}$ is abelian; thus G is transitive on Ω by Theorem 1.3.

Step 4: The final step Without loss of generality assume that G acts on Δ_1 as in (2) above and acts 2-transitively on Δ_2 . As G acts as in (2), we know G must be A_7 or S_7 with $|\Delta_1| = 21$ and $d_1 = 10$. Thus $d_2 \cdot |\Delta_2| = d_1 \cdot |\Delta_1| = 210$ by Corollary 2.3. As G is 2-transitive on Δ_2 , it follows that $d_2 = |\Delta_2| - 1$ and so $|\Delta_2| \cdot (|\Delta_2| - 1) = 210$. In particular, $|\Delta_2|$ must be a positive integer solution to the quadratic equation $x^2 - x - 210 = 0$. This polynomial factors as $x^2 - x - 210 = (x - 15)(x + 14)$ and so $|\Delta_2| = 15$ and $d_2 = 14$. Now $d_1 = 10$ and $d_2 = 14$ which are not coprime, contradicting Lemma 2.1. Hence G can not act 2 transitively on Δ_2 .

Thus we have proven the following theorem.

Theorem 3.1. *If G is a finite quasi-transitive permutation group on a finite set Ω , then G is transitive on Ω .*

References

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